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LETTER TO THE EDITOR

Young diagrams as Kronecker products of symmetric or antisymmetric components

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Abstract. A general relation is presented for expressing any Young diagram as an operator polynomial acting on the Kronecker product formed from either the rows or columns of the original diagram. This relation is used to derive branching rules $G \supset H$ for compact groups G and H of GL(N) which are well suited for numerical computations. The examples chosen are the $SU(3) \supset SO(3)$ and $SU(4) \supset SU(2) \times SU(2)$ reductions employed in studies of nuclear structure and the $SU(8) \supset SU(3)$ decomposition employed in studies of particle physics.

Since their introduction by Littlewood (1943a, b, 1950) over forty-five years ago, Young diagrams have served as an invaluable tool to the group theorist for the characterisation of general linear GL(N) representations. Through their use with the Littlewood rules, mathematical physicists have determined the reduction of Kronecker (outer) products of irreducible representations (irreps) belonging to compact groups within GL(N), and ascertained the branching rules from their irreps to those of their associated subgroups (Wybourne 1974). Although the Littlewood method has been employed extensively for the resolution of these problems, it is found to be cumbersome for more than the simplest cases. This has therefore led to the use of machine computation for generating many of the available tabulations (Wybourne 1964, Perez and Flores 1968, McKay and Patera 1981). Although the method is adaptable to computer coding, as is evidenced by an available program called SCHUR (Wybourne 1989) which provides Kronecker products and branching rule reductions for GL(N) and any of its subgroups, it is not well suited to direct coding into simple algorithms for large scale numerical computations. It is not surprising, therefore, that considerable effort has been expended into finding practical alternatives or simplifications to this conventional procedure.

One method presented by Braunschweig and Hecht (1978) exploits the simplicity gained in decomposing Young diagrams in terms of Kronecker products of their completely symmetric or antisymmetric components. With this method, simple matrices d are constructed via a recursive process, and their cofactors evaluated, to determine the decomposition of the Young diagram. In this letter, an equivalent but much simpler operator realisation of this method is derived which circumvents the need to construct

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the d matrices and leads to a direct expansion of the Young diagram. This realisation is a generalisation to operator form of another method given by Bardeen and Feenberg (1938) which is derived here in a much more transparent manner than in their original proof. The utility of this operator realisation is illustrated by deriving analytic results for branching rules which are easily transcribed into simple computer algorithms. The examples chosen are for physically relevant subgroup decompositions of two unitary groups employed in nuclear structure models and one used in particle physics for the description of a many-gluon system.

Recall that a Young diagram $\{f_1 f_2 \ldots f_N\} = \{f\}^N$ is a pattern associated with an ordered partition of integers $f_1 \geqslant f_2 \ldots \geqslant f_N \geqslant 0$, which is composed of f_1 squares in the first row, f_2 squares in the second row, etc. Its conjugate pattern $\{\tilde{f}\}^N$ is defined as the diagram obtained by the interchange of rows and columns of $\{f\}^N$, and is given by the expression

$$\{\tilde{f}\}^N = \{N^{f_N}(N-1)^{f_{N-1}-f_N} \ldots 1^{f_1-f_2}\}$$

where $i^{f_i-f_{i+1}}$ represents $f_i - f_{i+1}$ successive occurrences of rows with i squares in each.

Consider now the general problem of evaluating the Kronecker product of an arbitrary Young diagram $\{\tilde{f}\}^N$ with a totally antisymmetric one $\{\tilde{f}_{N+1}\}$, where $f_{N+1} \leqslant f_N$. Applying the Littlewood rules for outer product multiplication, the series of Young diagrams which results is the sum of those diagrams obtained from $\{\tilde{f}\}^{N+1}$ after extracting squares from the last $N+1$ column and adding them to the first N columns in all possible ways. Using θ_i to denote the operator which adds a square to column i, and θ_i^{-1} the operator which removes one, this statement is expressed mathematically as

$$\{\tilde{f}\}^N \otimes \{\tilde{f}_{N+1}\} = \sum_{k_N=0}^{f_{N+1}} \sum_{(i)}^{k_N} \prod_{j=1}^{N} \theta_j^{i_j} \theta_{N+1}^{-k_N} \{\tilde{f}\}^{N+1} \tag{1}$$

where the sum over $(i) = (i_1, i_2, \ldots, i_N)$ represents all the sets of positive integers i_j which satisfy $\Sigma_{j=1}^N i_j = k_N$, and only Young diagrams $\{\mu\}^{N+1}$ associated with ordered partitions $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{N+1} \geqslant 0$ are retained in the final result.

Commencing with this simple relation, it is possible to derive a simple operator realisation for any Young diagram as a sum of Kronecker products of totally antisymmetric components. Begin by expanding the k_N sum, and observe that the sum of operator products acting on $\{\tilde{f}\}^{N+1}$ is equivalent to the expression

$$\{\tilde{f}\} \otimes \{\tilde{f}_{N+1}\} = \left\{1 + \sum_{j_1=1}^{N} \theta_{j_1} \theta_{N+1}^{-1} + \sum_{j_1, j_2=1}^{N} \theta_{j_1} \theta_{j_2} \theta_{N+1}^{-2} + \ldots \right.$$
$$\left. + \sum_{j_1, \ldots, j_{f_{N+1}}=1}^{N} \theta_{j_1} \ldots \theta_{j_{f_{N+1}}} \theta_{N+1}^{-f_{N+1}} \right\} \{\tilde{f}\}^{N+1}. \tag{2}$$

Since applying θ_{N+1}^{-M} to $\{\tilde{f}\}^{N+1}$ yields zero whenever $M > f_{N+1}$, one realises the above sum of θ operator products can be extended to infinity, which allows the expression to be rearranged into the more compact form

$$\{\tilde{f}\}^N \otimes \{\tilde{f}_{N+1}\} = \{1 + \theta_1 \theta_{N+1}^{-1} + \theta_1^2 \theta_{N+1}^{-2} + \ldots\}\{1 + \theta_2 \theta_{N+1}^{-1} + \theta_2^2 \theta_{N+1}^{-2} + \ldots\}$$
$$\times \ldots \times \{1 + \theta_N \theta_{N+1}^{-1} + \theta_N^2 \theta_{N+1}^{-2} + \ldots\}\{\tilde{f}\}^{N+1}$$
$$= \prod_{j=1}^{N} \left[\sum_{l=0}^{\infty} (\theta_j \theta_{N+1}^{-1})^l \right] \{\tilde{f}\}^{N+1}. \tag{3}$$

Now, the operator expression in square brackets is simply a geometric series, and will yield unity when acted upon by $\Pi_{j=1}^{N}(1-\theta_j\theta_{N+1}^{-1})$. Thus, applying such an operator leads to the simple relation

$$\prod_{j=1}^{N}(1-\theta_j\theta_{N+1}^{-1})\{\tilde{f}\}^N\otimes\{\tilde{f}_{N+1}\}=\{\tilde{f}\}^{N+1} \tag{4}$$

from which one can obtain by induction the expression

$$\{\tilde{f}\}^N=\prod_{\substack{i,j=1\\ i<j}}^{N}(1-\theta_i\theta_j^{-1})\{\tilde{f}\}\otimes\ldots\otimes\{\tilde{f}_N\}. \tag{5}$$

A similar statement to (5) holds for totally symmetric Young diagrams on realising that an expression analogous to (1) can be written

$$\{f\}^N\otimes\{f_{N+1}\}=\sum_{k_N=0}^{f_{N+1}}\sum_{(i)}^{k_N}\prod_{j=1}^{N}\phi_j^{i_j}\phi_{N+1}^{-k_N}\{f\}^{N+1} \qquad f_{N+1}\leqslant f_N \tag{6}$$

where the ϕ_j operators here denote the addition of a square to the row i. Identical arguments to those outlined above then lead to the result

$$\{f\}^N=\prod_{\substack{i,j=1\\ i<j}}^{N}(1-\phi_i\phi_j^{-1})\{f_1\}\otimes\ldots\otimes\{f_N\}. \tag{7}$$

Relations (5) and (7) are the essential results contained within this letter. They state that any Young diagram may be expressed as a sum of Kronecker products of totally symmetric or antisymmetric diagrams, which are respectively determined by evaluating the action of operator polynomials on the initial Kronecker product formed from the rows or columns of the original diagram. In the resulting expansion, it is implicitly understood that only those products are retained which contain non-negative symmetric or antisymmetric diagrams.

These two derived relations may be viewed as much simpler operator realisations of expressions given previously by Braunschweig and Hecht (1978). However, in contrast to the former relations which required the recursive construction of simple matrices d, and the subsequent evaluation of cofactors for specific matrix elements d_{ij}, the expressions presented here enable the expansion of the $\{f\}^N$ Young diagram to be obtained immediately from the direct action of operator polynomials. This method is preferable when carrying out numerical computations, as it avoids the necessity of constructing an entire matrix d and then evaluating cofactors to determine the expansion coefficients.

An immediate example which illustrates the advantages of this operator formalism are the special cases of two-column and two-rowed diagrams considered by Braunschweig and Hecht (1978). Applying relations (5) and (7) one trivially retrieves the two quoted results

$$\{\widetilde{f_1f_2}\}=\{1^{f_1}\}\otimes\{1^{f_2}\}-\{1^{f_1+1}\}\otimes\{1^{f_2-1}\} \tag{8}$$

and

$$\{f_1f_2\}=\{f_1\}\otimes\{f_2\}-\{f_1+1\}\otimes\{f_2-1\}. \tag{9}$$

As previously outlined (Braunschweig and Hecht 1978), a very useful application for relations expressing Young diagrams as expansions of Kronecker products is the determination of irrep branching rules for group chains G ⊃ H, where G and H are subgroups of GL(N), whenever the irreps of H contained within either the totally symmetric or antisymmetric irreps of G are known. In these cases the calculation of the branching rule is reduced to the problem of evaluating irrep Kronecker products belonging to H and simplifying the resulting summation expansion. Furthermore, if a closed formula is known for the irrep Kronecker product reduction, then the branching rule expressions may be derived in analytic form, which yields useful equations for large scale numerical computations.

A first example, which is of relevance for SU(3) shell model calculations, is the branching rule of irreps in the SU(3) ⊃ SO(3) chain. Since all SU(3) diagrams are at most two-rowed, one obtains in the Elliott (1958) notation from (9) the well known Kronecker product result

$$(\lambda,\mu)=(\lambda+\mu,0)\otimes(\mu,0)-(\lambda+\mu+1,0)\otimes(\mu-1,0). \tag{10}$$

With substitution of the branching rule for multiplicity free SU(3) irreps

$$(\lambda,0)\downarrow\sum_{i=0}^{[\lambda/2]}[\lambda-2i] \tag{11}$$

where $[x]$ denotes the greatest integer function, i.e. the largest integer of x when x is positive and zero when it is negative†, one obtains on evaluation of the SO(3) Kronecker products $[\lambda+\mu-2j]\otimes[\mu-2i]$ plus $[\lambda+\mu+1-2j]\otimes[\mu-1-2i]$, the general reduction formula

$$(\lambda,\mu)\downarrow\sum_{j=0}^{[(\lambda+\mu)/2]}\sum_{i=0}^{[\mu/2]}\sum_{k=|\lambda-2j+2i|}^{\lambda+2\mu-2j-2i}[k]-\sum_{j=0}^{[(\lambda+\mu+1)/2]}\sum_{i=0}^{[(\mu-1)/2]}\sum_{k=|\lambda+2-2j+2i|}^{\lambda+2\mu-2j-2i}[k]. \tag{12}$$

It can be verified that this formula reproduces the already known relation (Racah 1949, Draayer *et al* 1968)

$$(\lambda,\mu)\downarrow\sum_{k=0}^{\lambda+\mu}n_{(\lambda,\mu)}(k)[k] \tag{13}$$

where the SU(3)↓SO(3) multiplicity function $n_{(\lambda\mu)}(k)$ is given by

$$n_{(\lambda,\mu)}(k)=\left[\frac{\lambda+\mu+2-k}{2}\right]-\left[\frac{\lambda+1-k}{2}\right]-\left[\frac{\mu+1-k}{2}\right].$$

Although this last expression (13) is more efficient for numerical computations, it is a special result that only applies to the SU(3)↓SO(3) case, while the former relation (12) was derived by a more direct method using (7), which is applicable to any general G↓H decomposition within GL(N) when the reduction for one-rowed representations of G are known along with closed formulae for irrep Kronecker products of H.

To illustrate that the method remains simple and direct for larger groups, consider next the example of the Wigner supermultiplet SU(4)↓SU(2) × SU(2) reduction employed for determining the spin-isospin content of nuclei (Hecht and Pang 1969). The conventional approach is to determine the branching rule through the group chain SU(4) ⊃ SO(4) ⊃ SU(2) × SU(2), and will be the one adopted here. Applying relation (7), the three-rowed SU(4) representation is first expanded in terms of Kronecker

† To remain consistent with standard notation the symbol [ ] is used to denote both SO(N) irreps and the greatest integer function. Its meaning as an irrep or function should be clear from the context in which it appears.

products of one-rowed representations:

$$\{f_1f_2f_3\} = \prod_{\substack{i,j=1\\ i<j}}^{3} (1-\phi_i\phi_j^{-1})\{f_1\}\otimes\{f_2\}\otimes\{f_3\}$$
$$= \{f_1\}\otimes\{f_2\}\otimes\{f_3\} + \{f_1+1\}\otimes\{f_2+1\}\otimes\{f_3-2\}$$
$$+\{f_1+2\}\otimes\{f_2-1\}\otimes\{f_3-1\} - \{f_1+1\}\otimes\{f_2-1\}\otimes\{f_3\}$$
$$-\{f_1\}\otimes\{f_2+1\}\otimes\{f_3-1\} - \{f_1+2\}\otimes\{f_2\}\otimes\{f_3-2\}. \tag{14}$$

For totally symmetric representations the SU(4)↓SO(4) reduction is trivially expressed as

$$\{f\}\downarrow \sum_{i=0}^{[f/2]} [f-2i]. \tag{15}$$

Insertion of the branching rule into the expansion for $\{f_1f_2f_3\}$ yields

$$\{f_1f_2f_3\}\downarrow \sum_{(i)=0}^{(\mathscr{P})} [p_1-2i_1]\otimes[p_2-2i_2]\otimes[p_3-2i_3]$$
$$-\sum_{(i)=0}^{(\mathscr{Q})} [q_1-2i_1]\otimes[q_2-2i_2]\otimes[q_3-2i_3] \tag{16}$$

where the summation over $(i)=(i_1, i_2, i_3)$ represents three independent sums from zero to the upper limits $(\mathscr{P})=([p_1/2],[p_2/2],[p_3/2])$ and $(\mathscr{Q})=([q_1/2],[q_2/2],[q_3/2])$, which occur for each of the three sets of integers

$$(p_1, p_2, p_3) = \{(f_1, f_2, f_3), (f_1+1, f_2+1, f_3-2), (f_1+2, f_2-1, f_3-1)\}$$

and

$$(q_1, q_2, q_3) = \{(f_1+1, f_2-1, f_3), (f_1, f_2+1, f_3-1), (f_1+2, f_2, f_3-2)\}$$

respectively, i.e.

$$\sum_{(i)=0}^{(\mathscr{P})} \equiv \sum_{i_1=0}^{[p_1/2]}\sum_{i_2=0}^{[p_2/2]}\sum_{i_3=0}^{[p_3/2]}.$$

Naturally, only summations with non-negative integer sets are retained in the expression.

These Kronecker products of SO(4) representations may now be reduced using the well known analytic expression (Wybourne 1974, p 239)

$$[ab]\otimes[cd]\downarrow \sum_{\alpha_1=0}^{a+b,c+d}\ \sum_{\alpha_2=0}^{a-b,c-d} [a+c-\alpha_1-\alpha_2, b+d-\alpha_1+\alpha_2] \tag{17}$$

for which $\Sigma_{\alpha=0}^{e,f}$ represents the sum of α from zero to $\min(e,f)$. With this product formula the SU(4) ↓ SO(4) branching rule is determined, after some minor algebra, to be given by

$$\{f_1f_2f_3\}\downarrow \sum_{(i)=0}^{(\mathscr{P})}\sum_{(\alpha)=0}^{(\bar{\mathscr{P}})}\left[\sum_{k=1}^{3}\bar{p}_k-\sum_{l=1}^{4}\alpha_l, \sum_{l=1}^{4}(-1)^l\alpha_l\right]$$
$$-\sum_{(i)=0}^{(\mathscr{Q})}\sum_{(\alpha)=0}^{(\bar{\mathscr{Q}})}\left[\sum_{k=1}^{3}\bar{q}_k-\sum_{l=1}^{4}\alpha_l, \sum_{l=1}^{4}(-1)^l\alpha_l\right]. \tag{18}$$

In this expression the summation over $(\alpha)=(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ represents four independent sums from zero to the upper limits

$(\bar{\mathscr{P}}) = (\min(\bar{p}_1, \bar{p}_2), \min(\bar{p}_1, \bar{p}_2), \min(\bar{p}_1+\bar{p}_2-2\alpha_1, \bar{p}_3), \min(\bar{p}_1+\bar{p}_2-2\alpha_2, \bar{p}_3))$

and

$(\bar{\mathscr{Q}}) = (\min(\bar{q}_1, \bar{q}_2), \min(\bar{q}_1, \bar{q}_2), \min(\bar{q}_1+\bar{q}_2-2\alpha_1, \bar{q}_3), \min(\bar{q}_1+\bar{q}_2-2\alpha_2, \bar{q}_3))$

respectively, where the simplifications $\bar{p}_k=p_k-2i_k$ and $\bar{q}_k=q_k-2i_k$, $k=1,2,3$ have been made.

Since $SU(2)\times SU(2)$ is the covering group of SO(4), the $SU(4)\downarrow SU(2)\times SU(2)$ relation is now trivially obtained from (17) by the $SO(4)\downarrow SU(2)\times SU(2)$ isomorphism

$$[f_1f_2]\downarrow(\tfrac{1}{2}(f_1+f_2),\tfrac{1}{2}(f_1-f_2))=(S,T). \tag{19}$$

Employing relations (18) and (19) one may easily code a simple computer algorithm which gives the $SU(4)\downarrow SU(2)\times SU(2)$ decomposition for any representation. Again it may be verified that these formulae reproduce the previously known relation (Racah 1949)

$$\{f_1f_2f_3\}\downarrow\sum_{S,T}n_{ST}(f_1f_2f_3)(S,T) \tag{20}$$

where the double sum ranges over $S, T=\frac{1}{2}(f_1+f_2-f_3), \frac{1}{2}(f_1+f_2-f_3)-1,\ldots,\frac{1}{2}$ or 0, and the $SU(4)\downarrow SU(2)\times SU(2)$ multiplicity function is given by

$$\begin{aligned}
n_{ST}(f_1f_2f_3)&=\omega_{ST}(f_1-f_3,f_2)-\omega_{ST}(f_1+1,f_2-f_3-1)-\omega_{ST}(f_1-f_2-1,f_3-1)\\
\omega_{ST}(ff')&=\varphi(f'+2-|S-T|)-\varphi(f'+1-S-T)+\varphi(S+T-f-1)\\
&\quad-\tfrac{1}{2}\varphi(S+T-|S-T|-f+f'+1)
\end{aligned} \tag{21}$$

$$\varphi(x)=\begin{cases}[x^2/4] & x>0\\ 0 & x\leqslant 0.\end{cases}$$

From a computational viewpoint this latter expression is more efficient than (18). However, as remarked in the former example, relation (18) was derived by a much simpler and more direct method which is applicable to any $GL(N)\supset G\supset H$ decomposition given two results: the reduction for one-rowed representations of G, and a closed formula for irrep Kronecker products of H. By contrast the derivations of (13) and (19) are not easily extended to larger groups. Moreover, results such as (12) and (18) can be easily coded into simple algorithms which, when used in conjunction with recently available routines for balanced binary tree data structures (Park and Draayer 1989), are still highly efficient for large scale numerical calculations.

The true merit of the method illustrated in the previous two examples becomes evident when considering a problem borrowed from quantum chromodynamics for a many-body system of spin-one gluons (Hess and Viollier 1986). For such a system the following chain of groups is found to be relevant

$$\begin{array}{ccccc}
\{N\} & & \{f_1f_2f_3\} & & \{f_1f_2f_3\}\\
U(24) & \supset & U(8) & \times & U(3)\\
 & & \cup\,\delta & & \cup\,\omega\\
 & & SU(3) & & SU(2)\\
 & & (\lambda,\mu) & & S
\end{array} \tag{22}$$

where U(8) and U(3) are associated with the eight gluon and three spin degrees of freedom, respectively. In the description of this system only the completely symmetric representation $\{N\}$ of U(24) is of importance. Since the direct product coupling of the U(8) and U(3) representations must yield the totally symmetric U(24) irrep $\{N\}$, the irreps of U(8) and U(3) are restricted to having the same Young diagram. This is at most three rows in general $\{f_1f_2f_3\}$ because of U(3). To exploit this group classification for the many-gluon system the (λ,μ) SU(3) and S SU(2) subirreps occurring within the unitary representations must be determined along with their respective multiplicity δ and ω.

The U(3) $\supset$ SU(2) reduction is already given by relation (12) as the gluons are spin one and the (λ, μ) SU(3) irrep is determined by $\lambda = f_1 - f_2$ plus $\mu = f_2 - f_3$. The U(8) $\supset$ SU(3) decomposition is given by equation (2.11) of Hess and Viollier (1986). Unfortunately, with the latter relation one needs to determine the coefficients appearing in the arbitrary inner product of two U(3) Young diagrams, which for most cases is cumbersome to obtain. Under restriction to a completely symmetric U(8) irrep, however, this formula simplifies to the following products of U(3) irreps

$$\{N\}\downarrow \sum_{g_1g_2g_3} \{g_1, g_2, g_3\}\otimes\{g_1-g_2, g_2-g_3, 0\} + \sum_{h_1h_2h_3} \{h_1, h_2, h_3\}\otimes\{h_1-h_2, h_2-h_3, 0\} \tag{23}$$

where the sums are constrained to integer values satisfying $\Sigma_{i=1}^{3} g_i = N$ and $\Sigma_{i=1}^{3} h_i = N-1$. These U(3) representations are trivially reduced to SU(3), whereupon the resulting outer products may be evaluated by applying the closed formula (O'Reilly 1982)

$$(s, t)\otimes(u, v) = \sum_{\alpha_1=0}^{v,s+t} \sum_{\alpha_2=0}^{t,u,s+t-\alpha_1} \sum_{\alpha_3=0,\alpha_2-t+\alpha_1}^{u-\alpha_2+\alpha_1,s} (s+u-\alpha_2-2\alpha_3+\alpha_1, t+v+\alpha_3-\alpha_2-2\alpha_1) \tag{24}$$

in which $\Sigma_{\alpha=a,b}^{c,d,e}$ is defined as the sum where α ranges from $\max(a, b)$ to $\min(c, d, e)$. With the benefit of these last two relations and (14) a complete decomposition of an arbitrary three-rowed irrep of U(8) in terms of SU(3) may be determined. As an example, the reduction of the {531} U(8) irrep is given in table 1.

Table 1. The U(8) $\supset$ SU(3) decomposition $\{531\}\downarrow(\lambda, \mu)$. The δ and dim respectively refer to the multiplicity and dimension of the corresponding SU(3) subirreps within {531}, which is of dimension 128 304.

(λ, μ)	δ	dim	(λ, μ)	δ	dim
(12, 0), (0, 12)	1	91	(11, 2), (2, 11)	2	270
(10, 4), (4, 10)	1	440	(10, 1), (1, 10)	7	143
(9, 3), (3, 9)	8	280	(9, 0), (0, 9)	12	55
(8, 5), (5, 8)	4	405	(8, 2), (2, 8)	26	162
(7, 7)	1	512	(7, 4), (4, 7)	23	260
(7, 1), (1, 7)	47	80	(6, 6)	12	343
(6, 3), (3, 6)	62	154	(6, 0), (0, 6)	45	28
(5, 5)	45	216	(5, 2), (2, 5)	104	81
(4, 4)	101	125	(4, 1), (1, 4)	108	35
(3, 3)	142	64	(3, 0), (0, 3)	58	10
(2, 2)	128	27	(1, 1)	63	8
(0, 0)	11	1			

It would not have been practical to obtain the tabulated result for {531} using the method outlined by Hess and Viollier (1986). In contrast, the procedure described here enables a simple computer algorithm to be developed for carrying out this reduction. To test the applicability of the program, the SU(3) decomposition for the {12, 10, 8} and {24, 20, 16} U(8) irreps were also carried out. Here we only quote the quantity of physical interest, namely the number of SU(3) colour singlets occurring within these irreps, 3960 and 737 719, respectively.

Clearly, there are numerous other examples which could be added, such as the $SU[(N+1)(N+2)/2] \supset SU(3)$ reduction employed for determining the available SU(3) irreps in the Nth nuclear shell (Draayer *et al* 1989). The ones enclosed within this letter should suffice to demonstrate the broad range of physical group reductions for which relations (5) and (7) may be effectively applied.

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